Mean-field approximation and a small parameter in turbulence theory

Victor Yakhot*
School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540
(Received 22 February 2000; revised manuscript received 1 September 2000; published 26 January 2001)

Numerical and physical experiments on two-dimensional (2D) turbulence show that the differences of transverse components of velocity field are well described by Gaussian statistics and Kolmogorov scaling exponents. In this case the dissipation fluctuations are irrelevant in the limit of small viscosity. In general, one can assume the existence of a critical space dimensionality $d=d_c$, at which the energy flux and all odd-order moments of velocity difference change sign and the dissipation fluctuations become dynamically unimportant. At $d<d_c$ the flow can be described by the ''mean-field theory,'' leading to the observed Gaussian statistics and Kolmogorov scaling of transverse velocity differences. It is shown that in the vicinity of $d=d_c$ the ratio of the relaxation and translation characteristic times decreases to zero, thus giving rise to a small parameter of the theory. The expressions for pressure and dissipation contributions to the exact equation for the generating function of transverse velocity differences are derived in the vicinity of $d=d_c$. The resulting equation describes experimental data on two-dimensional turbulence and demonstrates the onset of intermittency as $d-d_c>0$ and $r/L\to 0$ in three-dimensional flows in close agreement with experimental data. In addition, some exact relations between correlation functions of velocity differences are derived. It is also predicted that the single-point probability density function of transverse velocity components in developing as well as in the large-scale stabilized two-dimensional turbulence is a Gaussian.

DOI: 10.1103/PhysRevE.63.026307 PACS number(s): 47.27.Ak

I. INTRODUCTION

The role of the mean-field theories and Gaussian limits as starting points for understanding such important physical phenomena as superconductivity, superfluidity, critical point, naming just a few, can hardly be overestimated. These theories, usually based on remarkable physical intuition and insight, provided mathematical and intellectual foundations for investigation of much more difficult regimes in terms of deviations from the mean-field solutions. The most recent example is a theory of anomalous scaling in a model of a passive scalar, advected by a random velocity field, which was developed as an expansion in powers of small parameters characterizing deviations from the two Gaussian limits [1–3]. In a typical nonlinear system, a Gaussian limit corresponds to a weak coupling asymptotics and, as a consequence, to a ''normal,'' nonanomalous, scaling, which can often be obtained from a ''bare'' or linearized problem. A good example of this behavior is a fluid in thermodynamic equilibrium.

The large-Reynolds-number three-dimensional (3D) strong turbulence is characterized by an $O(1)$ energy flux $\mathcal{E}=\nu\overline{(\partial_i v_j)^2}$, which in many flows is $O(v_{\rm rms}^3/L)$, where L is an integral scale of turbulence. In the inertial range, where $k\ll k_d\to\infty$ or $r/L\ll 1$, the observed energy spectrum $E(k)$ is close to the one proposed by Kolmogorov and ''the probability density function of velocity increments'' $P(\Delta u)$ with $\Delta u=u(x+r)-u(x)$ is far from the Gaussian. Moreover, the experiments revealed a scaling law for the moments of velocity difference $S_{n,0}=\overline{(\Delta u)^n}\propto r^{\xi n}$ with the exponents ξ_n which cannot be obtained on dimensional grounds. This anomalous scaling and the very existence of the energy flux, resulting in the nonzero value of the third-order moment $S_{3,0}=\overline{(\Delta u)^3}\approx O(r)$, where $\mathbf{u}\cdot\mathbf{r}=ur$, imply a strongly non-Gaussian process and an obvious lack of the mean-field limit.

The situation may not be so grim, however: all odd-order moments of transverse velocity differences in both 2D and 3D flows $S_{0,2n+1}=\overline{[v(x+r)-v(x)]^{2n+1}}=0$ with $\mathbf{v}\cdot\mathbf{r}=\mathbf{0}$. This fact tells us that these components of velocity differences do not participate in the interscale energy transfer and there is no *a priori* reason for them not to obey Gaussian statistics in some limiting cases. This is indeed true in two-dimensional turbulence in the inverse cascade range where $l_f\ll r\ll L$ and l_f is a forcing scale.

Numerical and physical experiments on external-force-driven two-dimensional turbulence showed that the moments of transverse velocity differences and even-order moments of the longitudinal ones are very close to the Gaussian values and are characterized by the Kolmogorov scaling exponents $S_{0,2n}\propto S_{2n,0}\propto r^{\xi_{2n}}$ with $\xi_{2n}=2n/3$ [4–7]. The odd-order moments of longitudinal velocity differences are positive in a 2D flow, while they all are negative in (3A). This observation tells us that the most important distinction between two- and three-dimensional turbulence is in the dynamic role of the dissipation contributions: they are irrelevant in the two-dimensional inverse cascade range and are crucial for the small-scale dynamics of a three-dimensional flow where the forcing terms can be neglected. Thus, we can assume that it is the dissipation fluctuations that are responsible for both strong deviations from the Gaussian statistics and anomalous scaling in three-dimensional flows. This assumption is consistent with the observation of the close-to-Gaussian probability density of velocity differences in 3D turbulence at the

*On leave from Department of Aerospace and Mechanical Engineering, Boston University, Boston, MA 02215 scales $r\approx L$ where the integral scale L is defined as the one at which $S_{3,0}(L)=0$ and the intermittent dissipation fluctuations disappear [8]. It is clear that this range ($r\approx L$) is not characterized by well-defined scaling exponents.

As will become clear below, the forcing contribution to the equation for the probability density involves a factor $\mu \approx 1-\cos(k_f r)$. This means that at the scales $r \gg 1/k_f$ the parameter $\mu\approx 1$ while $\mu\approx (k_f r)^2$ when $k_f r \ll 1$. Thus, the forcing term must be important in the inertial range of two-dimensional turbulence ($r \gg l_f$) and is irrelevant in the 3D inertial range with the positive energy flux. This change happens at some space dimensionality $d=d_c$ at which the energy flux changes sign. The calculation of $d_c\approx 2.05$ was conducted by Frisch and Fourier [9] within the framework of a simple closure model. The more physically transparent calculation can be performed for the Navier-Stokes equations driven by a random force having an algebraically decaying spectrum in the inertial range [10], where a one-loop small-scale-elimination procedure gives a correction to the bare viscosity,

$$\delta\nu=\nu\frac{d^2-d-\epsilon}{2d(d+2)}\mathrm{Re}^2,$$

where Re is a properly defined Reynolds number corresponding to the eliminated small-scale velocity fluctuations and ϵ is a parameter characterizing the forcing function. In case of Kolmogorov turbulence $\epsilon\approx 4$. This relation shows that the role the small scales play in turbulence dynamics depends on the space dimensionality d: the correction to viscosity is positive when $d>d_c(\epsilon)$ and it changes sign at $d=d_c+$. Physically, this means that the small-scale velocity fluctuations take energy from the large-scales motions (direct energy cascade from large-to-small structures) at $d>d_c$, while at $d<d_c$ they excite the large-scale motions (spend their energy) giving rise to the inverse energy cascade. For $\epsilon=4$ the critical dimensionality $d_c\approx 2.56$. The correct value of d_c is not too important: what is crucial for the theory presented below is that the critical dimensionality, at which the flux changes its sign, exists. It will be shown below that $d-d_c\to 0$ is a small parameter of the theory enabling one to calculate an expression for the dissipation anomaly in a form resembling the Kolmogorov refined similarity hypothesis.

Since in 2D the moments $S_{0,2n}$ show Kolmogorov scaling, the Gaussian statistics of transverse velocity differences cannot correspond to the weak coupling limit. This problem was considered in Ref. [11] where, following Polyakov [12], the equation for the generating function for the problem of the Navier-Stokes turbulence was introduced. An unusual symmetry of this equation enabled one to show that the solution was consistent with both Kolmogorov scaling and Gaussian statistics. In this work a more detailed theory of two-dimensional turbulence is presented and the generalization to three-dimensional flows is considered. The main result of the paper is a model demonstrating how the deviations from the ''normal scaling'' and Gaussian statistics appear in 3D when the strength of the dissipation term deviates from zero and the ''scale'' parameter $\epsilon_0=1-r/L$ deviate from zero.

This paper is organized as follows. In Sec. II the equation for the generating function, derived in [11], is introduced. Some exact relations between velocity structure functions, following from this equation, are derived in Sec. III. The connection between scaling exponents of the moments of velocity differences and their amplitudes is established in Sec. IV. The mean-field derivation of the pressure term is given in Sec. V which is used to obtain a Gaussian probability density function (PDF) in the two-dimensional flow in Sec. VI. In Sec. VII the small parameter of the theory is identified and used to derive the expression for the dissipation contributions to the equation for the PDF. Section VIII is devoted to a solution of the equation and a demonstration of how anomalous scaling and deviations from Gaussian statistics emerge from the theory. Conclusive remarks are presented in Sec. IX.

II. EQUATION FOR THE GENERATING FUNCTION

The equations of motion are (density $\rho\equiv 1$)

$$\partial_t v_i + v_j\partial_j v_i = -\partial_i p + \nu\nabla^2 v_i + f_i, \quad \partial_i v_i = 0 \tag{1}$$

where $\mathbf{f}$ is a forcing function responsible for the kinetic energy production and in a statistically steady state the mean pumping rate $P=\overline{\mathbf{f}\cdot\mathbf{v}}$. In what follows we will be mainly interested in the probability density function of the two-point velocity difference $\mathbf{U}=\mathbf{u}(\mathbf{x}')-\mathbf{u}(\mathbf{x})\equiv\Delta\mathbf{u}$. The generating function is $Z=\langle\exp(\lambda\cdot\mathbf{U})\rangle$. The equation for the generating function of velocity differences corresponding to Eq. (1) is

$$\frac{\partial Z}{\partial t}+\frac{\partial^2 Z}{\partial\lambda_\mu\partial r_\mu}=I_f+I_p+D, \tag{2}$$

with

$$I_f=\langle\lambda\cdot\nabla\mathbf{f}e^{\lambda\cdot\Delta\mathbf{u}}\rangle, \tag{3}$$

$$I_p=-\lambda\cdot\langle e^{\lambda\cdot\Delta\mathbf{u}}\Delta(\nabla p)\rangle\equiv-\lambda\cdot\langle e^{\lambda\cdot\mathbf{U}}[\nabla_2 p(x_2)-\nabla_1 p(x_1)]\rangle \tag{4}$$

and

$$D=\nu\lambda\cdot\langle[\nabla_2^2\mathbf{v}(\mathbf{x}_2)-\nabla_1^2\mathbf{v}(\mathbf{x}_1)]e^{\lambda\cdot U}\rangle. \tag{5}$$

The most interesting and surprising feature of Eq. (2) is the fact that, unlike in the problem of Burgers turbulence [12], the advective contributions are represented here in a closed form. This means that the theory, developed below, is free from the troubles related to the Galilean invariance, haunting all schemes, based on renormalized perturbation expansions in powers of the Reynolds number. To completely close the problem the expressions for I_p and D are needed. Equations (2) and (3) formulate the turbulence theory in terms of ''only'' two unknowns I_p and D. The Kolmogorov refined similarity hypothesis stating that $(\Delta u)^3=\phi\mathcal{E}_r r$, where ϕ is a scale-independent random process and $\mathcal{E}_r$ is the dissipation rate averaged over a ball of radius r around point x, can be a promising starting point to a closure for the dissipation term D. This will be done below. The pressure term in Eqs. (2) and (3) is also of a very specific and rather limited nature: all we have to know is the correlation functions $\langle U_i U_j \cdots U_m \Delta\nabla p\rangle$. Thus, the definite targets needed for derivation of the closed equation for Z functions are well defined.

The generating function can depend only on three variables: $\eta_1 = r$; $\eta_2 = \lambda\cdot\mathbf{r}/r \equiv \lambda\cos(\theta)$; $\eta_3 = \sqrt{\lambda^2 - \eta_2^2}$; and

$$Z_t + \left[\partial_{\eta_1}\partial_{\eta_2} + \frac{d-1}{r}\partial_{\eta_2} + \frac{\eta_3}{r}\partial_{\eta_2}\partial_{\eta_3} + \frac{(2-d)\eta_2}{r\eta_3}\partial_{\eta_3} - \frac{\eta_2}{r}\partial^2_{\eta_3}\right]Z = I_f + I_p + D. \tag{6}$$

Below we will often use $\partial_{\eta_i} \equiv \partial_i$. The functions I_p, I_f, and D are easily extracted from the above definitions. Let us denote $\Delta u \equiv U$ and $\Delta v \equiv V$. In the new variables the generating function can be represented as

$$Z = \langle e^{\eta_2 \Delta u + \eta_3 \Delta v}\rangle \equiv \langle e^{\eta_2 U + \eta_3 V}\rangle$$

with the mean dissipation rate $\mathcal{E}$ defined by $\overline{\nu(\partial_x u)^2} = (1/d)\mathcal{E}$. Any correlation function is thus

$$S_{n,m} \equiv \langle U^n V^m\rangle = \partial_2^n \partial_3^m Z(\eta_2 = \eta_3 = 0, r).$$

III. RELATIONS BETWEEN MOMENTS OF VELOCITY DIFFERENCE

Let us discuss some direct consequences of Eqs. (1)–(6). The Navier-Stokes equations are invariant under transformation: $v \rightarrow -v$ and $y \rightarrow -y$. That is why $\langle[\partial_y p(0) - \partial_y p(r)] \times (\Delta v)^m\rangle \neq 0$ if $m = 2n+1$ with $n > 1$ and is equal to zero if $m = 2n$. It is also clear from the symmetry that $\langle \partial_x p U^{2n}\rangle$ is an odd function of r. It follows from the Navier-Stokes equations that $\nu\langle(\nabla^2 U)(\Delta u)^{2n}\rangle = 0$ when $\nu \rightarrow 0$ and $\langle(\nabla^2 V)V^{2n}\rangle = 0$. The first relation is proved in the following way: a single-point correlation function

$$\langle(\nabla^2 u)u^{2n}\rangle = 0,$$

due to the symmetry of the problem. Thus $\langle\nabla^2 u(\Delta u)^{2n}\rangle \propto r|r|^\gamma$ with $1+\gamma > 0$, meaning that multiplied by $\nu \rightarrow 0$, this expression tends to zero. This result is a consequence of a simple observation that the above expression can be represented as the sum of single-point contributions (equal to zero) and the functions of r with the displacement r in the inertial range. This proves the above relation.

Multiplying Eq. (6) by η_3 and applying $\partial_3 \partial_2^{2n-1}$ to the resulting equation gives as $\eta_2 = \eta_3 \rightarrow 0$

$$\frac{\partial S_{2n,0}}{\partial r} + \frac{d-1}{r}S_{2n,0} - \frac{(d-1)(2n-1)}{r}S_{2n-2,2} = -(2n-1)\langle \Delta p_x (\Delta u)^{2n-2}\rangle + P[1 - \cos(r/l_f)]a_{n,d}S_{2n-3,0}, \tag{7}$$

where $a_{n,d} = 2(2n-1)(2n-2)/d$. Due to the symmetry of the problem dissipation terms do not contribute to this relation.

In a three-dimensional case (3D) the small-scale ($r/l_f \ll 1$) contribution to the right side of Eq. (7) is $O(r^2)$ and can be neglected. It is, however, $O(1)$ in two-dimensional turbulence in the inverse cascade range where $r/l_f \gg 1$. For $n=1$ Eq. (7) gives a well-known incompressibility relation [13,14]

$$\frac{\partial S_{2,0}}{\partial r} + \frac{d-1}{r}S_{2,0} = \frac{d-1}{r}S_{0,2}. \tag{8}$$

Multiplying Eq. (6) by η_3 with subsequent $\partial_3 \partial_2^2$ leads, after setting $\eta_2 = \eta_3 = 0$ as $\nu \rightarrow 0$, to

$$\frac{\partial S_{3,0}}{\partial r} + \frac{d-1}{r}S_{3,0} - 2\frac{d-1}{r}S_{1,2} = (-1)^d \frac{4}{d}P, \tag{9}$$

where $d=2$ and 3. Applying $\partial_3^3 \eta_3$ to Eq. (6) gives, as $\nu \rightarrow 0$,

$$\frac{\partial S_{1,2}}{\partial r} + \frac{d+1}{r}S_{1,2} = (-1)^d \frac{4}{d}P. \tag{10}$$

Substituting this into Eq. (9) yields a well-known Kolmogorov relation

$$S_{3,0} \equiv \overline{(\Delta u)^3} = (-1)^d \frac{12}{d(d+2)}Pr. \tag{11}$$

For $2n=4$ the relation (7) reads

$$\frac{\partial S_{4,0}}{\partial r} + \frac{d-1}{r}S_{4,0} = \frac{3(d-1)}{r}S_{2,2} - 3\langle(\Delta p_x)(\Delta u)^2\rangle.$$

This relation is correct in the case of incompressible and isotropic turbulence when $\nu \rightarrow 0$, including the limit $r \rightarrow L$ where velocity fluctuations obey close-to-Gaussian statistics (see below). There, since $\langle\nabla p\rangle = 0$, the pressure contribution is negligible. In the dissipation range

$$\langle \Delta p_x (\delta u)^2\rangle \approx r^3 \langle p_{xx} u_x^2\rangle \ll r^3 \langle u_x^4\rangle$$

since $\langle p_{xx}\rangle = 0$. This leads us to the conclusion that the pressure contribution, at least numerically, is small in the inertial range, too. This gives

$$\frac{\partial S_{4,0}}{\partial r} + \frac{d-1}{r}S_{4,0} \approx \frac{3(d-1)}{r}S_{2,2}. \tag{12}$$

In two-dimensional turbulence ($d=2$), where the $O(d-2)$ contribution to Eq. (6) is zero, one can neglect the dissipation term D in the inverse cascade range and derive

$$\frac{\partial S_{1,2n}}{\partial r} + \frac{1+2n}{r}S_{1,2n} = n(2n-1)PS_{0,2n-2} - 2n\langle \mathcal{P}_{yv}(\Delta v)^{2n-1}\rangle, \tag{13}$$

where $\mathcal{P}_{yv} \equiv \partial_y p(x+r) - \partial_y p(x)$. Another interesting relation, valid in 2D is obtained from Eq. (6) by differentiating once over η_2 and $2n$ times over η_3,

$$\frac{\partial S_{2,2n}}{\partial r}+\frac{1+2n}{r}S_{2,2n}=\frac{S_{0,2n+2}}{r}-2n\langle \mathcal{P}_{yv}\Delta u(\Delta v)^{2n-1}\rangle +n(2n-1)PS_{1,2n-2}. \quad (14)$$

In the direct cascade range, where the forcing contribution is $O(r^2)\to 0$, the relation (14) for an arbitrary dimensionality d reads

$$\frac{\partial S_{2,2n}}{\partial r}+\frac{d-1+2n}{r}S_{2,2n}=\frac{2n+d-1}{2n+1}\frac{S_{0,2n+2}}{r}-2n\langle \mathcal{P}_{yv}\Delta u(\Delta v)^{2n-1}\rangle. \quad (15)$$

We would like to stress the difference between Eq. (13) and Eqs. (14) and (15). The relation (13) is for the odd-order structure functions with the dissipation term irrelevant in 2D only. On the other hand, the expressions (14) and (15) involve even-order moments with the dissipation contributions equal to zero when $\nu\to 0$ for an arbitrary space dimensionality. The relations derived in this section are of importance since they enable one to obtain information about pressure-velocity correlation functions in turbulent flows by experimentally measuring the combinations of velocity structure functions. Untill recently this information was unavailable.

Now, let us multiply Eq. (6) by η_3, differentiate once over η_2 and three times over η_3. This gives

$$\frac{\partial S_{2,2}}{\partial r}+\frac{d+1}{r}S_{2,2}=\frac{d+1}{3r}S_{0,4}-2\overline{\mathcal{P}_{yv}\Delta u\Delta v}. \quad (16)$$

This relation is correct since $\nu\overline{\nabla^2 v\Delta u\Delta v}=\nu\overline{\nabla^2 u(\Delta v)^2}=0$ when $\nu\to 0$.

2D simulations of Boffetta, Celani, and Vergassola. To achieve a true steady state these authors [7] conducted a series of very accurate simulations of the problem (1) with the large-scale dissipation term $D_L=-\alpha\mathbf{v}$ in the right side (6). The moments of transverse velocity differences, reported in this paper $(n\geqslant 2)$, were very close to their Gaussian values. It is clear that this term introduces $-\alpha\lambda_\mu(\partial Z/\partial\lambda_\mu)$ into the right side of Eq. (6) which is small in the inertial range where the nonlinearity is large. One has to be careful though, with the dangerous interval $\Delta v\to 0$ where the linear term is not small. We expect that the negative-order structure functions with $-1<n<0$ strongly depend on the functional shape of the otherwise irrelevant large-scale dissipation term. The same can be predicted for various conditional expectation values of dynamical variables, like pressure gradients and dissipation terms, for the fixed values of $\Delta v\Delta u$: near the origin where Δv and Δu are very small, the artificially introduced linear contributions to the Navier-Stokes equations dominates, producing large and nonuniversal deviations from the universal functions characterizing inertial range. For example, with the addition of the linear dissipation, relation (14) reads

$$\frac{\partial S_{2,2n}}{\partial r}+\frac{1+2n}{r}S_{2,2n}=\frac{S_{0,2n+2}}{r}-(2n+1)\alpha S_{1,2n} -2n\langle \mathcal{P}_{yv}\Delta u(\Delta v)^{2n-1}\rangle +n(2n-1)PS_{1,2n-2}, \quad (17)$$

modifying the balance (pressure contribution) in the range of the small product $\Delta u\Delta v$ or $r/L\approx 1$. In the interval where $\Delta u\Delta v$ is not small $(r\to 0)$, the linear terms are small and can be neglected. The results obtained in this section can also be derived with

$$Z=e^{\sqrt{d-1}\,\eta_3 V+\eta_2 U} \quad (18)$$

with the properly defined moments of V and U.

IV. ASYMPTOTIC VALUES OF EXPONENTS IN THREE-DIMENSIONAL FLOWS

In the case of intermittent turbulence $S_{m,n}=A_{m,n}r^{\xi_{m,n}}$ with the "anomalous" scaling exponents which cannot be obtained on dimensional grounds. We can see that in the inertial range of a three-dimensional flow $(r/l_f\ll 1)$ the right side of Eq. (7) is negligible and, as a result, for $n>1$, $\xi_{2n,0}=\xi_{2n-2,2}\equiv\xi_{2n}<2n/3$. Substituting this into Eq. (7) gives immediately

$$\xi_{2n}=(d-1)\left[(2n-1)\frac{A_{2n-2,2}}{A_{2n,0}}-1\right]. \quad (19)$$

Let us introduce the probability density functions $P(U)$ and $q(V|U)$ via

$$S_{2n,0}=\overline{U^{2n}}=\int P(U)U^{2n}dU \quad (20)$$

and

$$S_{2n-2,2}=\overline{U^{2n-2}V^2}=\int P(U)U^{2n-2}V^2q(V|U)dUdV, \quad (21)$$

where $q(V|U)$ is the conditional PDF of V for a fixed value of U. It is clear that $q(V|U)=q(-V,U)$, so that all odd-order moments of V are equal to zero. This expression can be rewritten as

$$S_{2n-2,2}=\overline{U^{2n-2}V^2}=\int P(U)U^{2n-2}Q_2(U)dU, \quad (22)$$

where Q_2 is a conditional expectation value of V^2 for a fixed value of U,

$$Q_2(U)=\int V^2q(V|U)dV. \quad (23)$$

It follows from Eq. (19) that the linear limit $\xi_{2n}\propto n(n\to\infty)$ is achieved only when the amplitudes $A_{2n,0}\approx A_{2n-2,2}$ which is possible only if $Q_2\propto U^2$. This seems rather improbable in the limit $U\to\infty(n\to\infty)$. As a result, the linear regime $\xi_{2n}\propto n$ is equally improbable. Saturation of exponents $\xi_{2n}\to\xi_\infty=\text{const}$ as $n\to\infty$ is possible on a rather wide class of probability densities. For example,

$$P(U)\to -A\frac{1}{U}\frac{\partial}{\partial U}P(U)Q_2(U) \tag{24}$$

where $A>0$ is a constant. Then,

$$S_{2n,0}=(2n-1)A\int P(U)Q_2(U)U^{2n-2}dU$$
$$=(2n-1)AS_{2n-2,2}. \tag{25}$$

Substituting this result into the expression for ξ_{2n} gives $\xi_{2n}\to(d-1)(A-1)=\text{const}$. The relation (24) defines the large-U asymptotics of the PDF $P(U)$ in terms of $Q_2(U)$,

$$P(U)\propto\frac{1}{Q_2(U)}\exp\left(-A\int^U\frac{udu}{Q_2(u)}\right). \tag{26}$$

If $Q_2(U)\to U^\beta$ then, assuming the existence of all moments, it follows from Eq. (25) that $\beta<2$. The ''log-normal'' PDF $P(U)$ corresponds to $Q_2(U)\propto U^2/2\log(U)$. The expression (24) also gives in the limit of large n

$$S_{2n+1,0}=\overline{U^{2n+1}}=2nA\overline{V^2U^{2n-1}}=2nAS_{2n-1,2}. \tag{27}$$

V. PRESSURE CONTRIBUTIONS

Due to the symmetries of the Navier-Stokes equations, neither pressure nor dissipation terms contributed to the expressions (7)–(12). To proceed further we have to evaluate I_p and D. First of all we see from Eq. (12) that $\xi_{4,0}=\xi_{2,2}$. Let us assume that in the inertial range $S_{4,0}=A_{4,0}r^{\xi_{4,0}}$, $S_{0,4}=A_{0,4}r^{\xi_{0,4}}$, and $S_{2,2}=A_{2,2}r^{\xi_{2,2}}$. Then, it is clear from Eqs. (12) and (16) that neglecting the pressure contribution to Eq. (16) gives $\xi_{4,0}=\xi_{0,4}$.

$d=2$. It will be shown in Sec. VI that in 2D the even-order moments of velocity differences are very close to the Gaussian ones and all exponents are close to the $K41$ values $\xi_n=n/3$. Then, $A_{4,0}=3A_{2,0}^2$ and $A_{0,4}=3A_{0,2}^2$. It follows from Eq. (12) that $A_{2,2}=7/9A_{4,0}$. From Eq. (18) when $d=2$ we have the amplitudes $5/3A_{2,0}=A_{0,2}$, we conclude that without the pressure contribution, Eqs. (12) and (16) are incompatible.

Following [15] we introduce a conditional expectation value of the pressure gradient difference for a fixed value of Δu, Δv, and r,

$$\langle\partial_y p(x+r)-\partial_y p(x)|\Delta u,\Delta v,r\rangle\approx\sum_{m,n}\kappa_{m,n}(r)(\Delta u)^m(\Delta v)^n, \tag{28}$$

where the functions $\kappa_{m,n}(r)$ ensure proper dimensionality of the corresponding correlation functions. The above expression explicitly assumes the existence of an expansion of the conditional expectation value (28). In general, this may not be true due to various singularities such as the ones arising in the dissipation contributions (see below). Since the pressure term involves only one spacial derivative, the ultraviolet singularity cannot appear. The infrared singularity is not there at least in 2D where the integral scale is time dependent (see below). Keeping only the first two terms of the expansion (28) produces a model for the pressure contributions

$$\langle\partial_y p(x+r)-\partial_y p(x)|\Delta y\Delta v\rangle\approx -h\frac{\Delta u\Delta v}{r}-b\frac{\Delta v}{(Pr)^{2/3}}. \tag{29}$$

Since in an incompressible and homogeneous flow $\overline{\Delta u\mathcal{P}_{xu}}=\overline{\Delta v\mathcal{P}_{yv}}=0$, the coefficients h and b are related as

$$-hS_{1,2}=bS_{0,2}(Pr)^{1/3}. \tag{30}$$

Limiting the expansion of a conditional expectation value by the first terms resembles Landau's theory of critical phenomena, well describing experimental data in a certain range of parameters variation. We will show below that in the case of turbulence this approximation gives the results which are in agreement with the data. This may be a consequence of the fact that $A_{0,4}\approx A_{4,0}\approx A_{2,2}=O(1)$.

With $\xi_n=n/3$ it follows from Eqs. (7), (13)–(15), and (29) that when $d=2$ and $n\to\infty$,

$$\left(\frac{2n(4-3h)}{3}+1\right)\frac{S_{1,2n}}{r}=n(2n-1)PS_{0,2n-2}+b\frac{2nS_{0,2n}}{(Pr)^{2/3}}$$

and

$$\left(\frac{2n(4-3h)}{3}+1\right)\frac{S_{2,2n}}{r}=\frac{S_{0,2n+2}}{r}+n(2n-1)PS_{1,2n-2}$$
$$+b\frac{2nS_{1,2n}}{(Pr)^{2/3}}.$$

In the limit $n\to\infty$, by assuming that $S_{1,2n}\approx nS_{1,2n-2}$, one derives readily

$$PrnS_{0,2n-2}\approx\frac{S_{2,2n+2}}{Prn}\approx\frac{1}{Prn^2}S_{0,2n+4}, \tag{31}$$

which is consistent with the Gaussian PDF $P(\Delta v)$ as $\Delta v\to\infty$. Thus, the relation (29) implies the Gaussian tails of the probability density. It is clear that due to the finite energy flux and relations (10) and (11), two-dimensional turbulence cannot be a Gaussian process. All the relation (31) can tell us is that the even-order moments with $n\gg 1$, described by Eq. (31), can be close to the Gaussian values. It will be shown below that transverse velocity differences, not directly involved in the interscale energy transfer, can obey Gaussian statistics. It is clear from Eq. (17) that the model (29) for the pressure contributions is wrong when the linear dissipation terms are added to the Navier-Stokes equations. In the limit of small $\Delta u\Delta v$ the balance is achieved when

$$\langle\mathcal{P}_{yv}|\Delta u,\Delta v,r\rangle\approx -h\frac{\Delta u\Delta v}{r}-\left(\alpha+\frac{b}{(Pr)^{2/3}}\right)\Delta v,$$

which differs from Eq. (29) in the range of small $\Delta u\Delta v$.

Three dimensional. In the intermittent three-dimensional turbulence $\xi_{2n}<\xi_{2n+1}$. This produces strong restrictions on the structure of the pressure contributions to Eq. (6). Let us assume that $\xi_{2n,0}=\xi_{2,2n-2}=\xi_{2n-2,2}$. Then, it is clear from Eq. (15) that the first term of expansion (28) has all right properties. The relation (15), involving the r derivatives, is valid for an arbitrary value of n and that is why any additional term of expansion (28) must not only depend on a proper power of n but the functions $\kappa_{m,n}(r)$ must also reflect nontrivial dimensionalities caused by the anomalous scaling exponents ξ_n. Since original Navier-Stokes equations do not involve noninteger powers of r, this possibility seems quite bizzarre. In what follows we will adopt the pressure model (29) in the three-dimensional case also.

VI. TWO-DIMENSIONAL TURBULENCE

Now we are interested in the case of the two-dimensional turbulence in the inverse cascade range. If a two-dimensional (2D) fluid is stirred by a random (or nonrandom) forcing acting on a scale $l_f=1/k_f$, the produced energy is spent on the creation of the large-scale ($l>l_f$) flow which cannot be dissipated in the limit of a large Reynolds number as $\nu\to 0$. This is a direct and most important consequence of an additional, entrophy conservation law, characteristic of two-dimensional hydrodynamics [16]. As a result, the dissipation term is irrelevant in the inverse cascade range and we set $D=0$ in Eq. (6) and hope that in two dimensions the situation is greatly simplified. This hope is supported by recent numerical and physical experiments [4–7] showing that as long as the integral scale $L_i\propto t^{3/2}$ is much smaller than the size of the system, the velocity field at the scales $L_i\gg l\gg l_f$ is a stationary close-to-Gaussian process characterized by the structure functions with the Kolmogorov exponents $\xi_n=n/3$. In a recent paper Boffetta, Celani, and Vergassola [7] reported the results of very accurate numerical simulations of two-dimensional turbulence generated by a random force. No deviations from Gaussian statistics of transverse velocity differences as well as from the Kolmogorov scaling $\xi_n=n/3$ were detected.

The pressure gradient $\partial_y p=\partial_y\partial_i\partial_j\partial^{-2}\Delta v_i\Delta v_j$ and the difficulty in calculating I_p is in the integral over the entire space defined by the inverse Laplacian ∂^{-2}. The huge simplification, valid in 2D, comes from the fact that all contributions to the left side of Eq. (6) as well as I_f are independent on time. This means that the integrals involved in the pressure terms cannot be infrared divergent since in a two-dimensional flow $L=L(t)\propto t^{3/2}$. We also have that $I_p\to\alpha^2 I_p$ when $U,V\to\alpha U;\alpha V$. Based on this and taking into account that $\langle(\Delta v)^{2n+1}(u_x^2+v_y^2+u_y v_x)\rangle=0$ we, in the limit $\eta_2\to 0$, adopt a low-order model (29) giving

$$I_p=\left[h\frac{\partial^2}{\partial\eta_2\partial\eta_3}+b\frac{\eta_3}{(Pr)^{2/3}}\partial_3\right]Z(\eta_2=0,\eta_3,r). \quad (32)$$

In two dimensions the relation (32) combined with Eq. (6) in the limit $\eta_2\to 0$ gives a closed equation for the moments of transverse velocity differences in 2D turbulence.

Substituting Eqs. (32) and (29) into Eq. (6) and, based on Eqs. (9)–(11), seeking a solution as $\eta_2\to 0$ as

$$Z(\eta_2,\eta_3,r)\approx Z_3(\eta_3,r)\varphi(\eta_2 r^{1/3},\eta_3)$$
$$\approx Z_3(\eta_3,r)\exp[\tfrac{1}{2}A_{2,0}(\eta_2 Pr^{1/3})^2]$$
$$\times\left(1+\tfrac{1}{2}A_{1,2}\eta_3^2\eta_2(Pr)+\frac{\eta_2^3}{4}+\cdots\right), \quad (33)$$

where ($A_{1,2}=1/2$), gives

$$\left[\partial_r+\frac{1}{r}+\frac{1-h}{r}\eta_3\partial_3\right]\frac{A_{1,2}Pr}{2}\eta_3^2 Z_3=2P\eta_3^2 Z_3+\frac{b\eta_3}{r^{2/3}}\partial_3 Z_3. \quad (34)$$

Setting $Z_3=Z_3(\eta_3 r^{1/3})\equiv Z_3(X)$ and $h=4/3$ one derives, using the relation (30) ($b=-2/3A_{0,2}$),

$$2A_{0,2}XZ_3=\partial_X Z_3,$$

corresponding to a Gaussian solution with the correct width $A_{0,2}$. This fact serves as a consistency check that the Gaussian is a solution for the PDF of transverse velocity differences. Equation (34) defines a probability density function corresponding to the finite moments $S_{2m,2n}(r)$ only when $h=4/3$. This situation resembles Polyakov's theory of Burgers turbulence [12] reduced to an eigenvalue problem with a single eigenvalue corresponding to the PDF which is positive in the entire interval.

Having these exact results and keeping in mind Eq. (33) one can integrate the equation over η_2 from $-i\infty$ to 0 to obtain

$$\frac{\partial Z_3}{\partial r}+3(1-h-b)\frac{\eta_3}{r}\frac{\partial Z_3}{\partial\eta_3}=\frac{2P}{(Pr)^{1/3}}\eta_3^2 Z_3 \quad (35)$$

which valid as long as

$$\frac{(\eta_3 r^{1/3})^2}{8A_{2,0}^2}\ll 1.$$

This constraint is an artifact of an approximate relation (33). As will be shown below Eq. (35) gives an exact Gaussian solution and thus is valid beyond the above interval. This result is obtained by choosing the integration function $\Psi(\eta_3,r)$ to compensate the $O(Z/r)$ term violating the normalizability constraint $Z(0,0,r)=1$. The solution to Eq. (35) is

$$Z_3=\exp[\gamma\eta_3^2(Pr)^{2/3}] \quad (36)$$

with the parameter $\gamma=3/[3(1-h-b)+1]\propto A_{0,2}$ defining the width of the Gaussian.

The first-order differential equation (35) for the generating function differences implies the underlying linear Langevin dynamics of transverse velocity differences. It is important that this equation in nonlocal in physical space but local in the Fourier one. The effective forcing, corresponding to the right-side of Eq. (35), is nonlocal and solution dependent.

To evaluate the single-point probability density that corresponds to velocity differences in the limit of large displacements r, we notice that the energy flux is not equal to zero only at $r \ll L$. At the distances $r \gg L(t)$ the zero value of the energy flux and symmetrization of the probability density $[P(\Delta u,L)=P(-\Delta u,L)]$ can be achieved only when the pressure contribution to Eq. (6) compensates the advective terms. As a result, since $D=0$, we have

$$Z_t = 2P\,\eta_3^2 Z.$$

Seeking a solution as $Z=Z(\eta_3\sqrt{t})\equiv Z[\eta_3 v_{rms}(t)]$ gives a Gaussian result

$$Z=e^{\eta_3^2 u_{rms}^2(t)}.$$

A similar outcome is obtained for the case investigated in [7]. When turbulence is stabilized at the large scales by an artificially introduced friction, the resulting equation is

$$\eta_3 \partial_3 Z = 2P\,\eta_3^2 Z,$$

which also leads to the Gaussian PDF. To conclude this section we would like to discuss the physical meaning of the integral scale L. The integral scale of turbulence is a scale at which the flux decreases to zero [18] and at which $S_{3,0}(L)=0$.

VII. SMALL PARAMETER IN TURBULENCE THEORY IN THREE DIMENSIONS

Three-dimensional turbulence is a notoriously difficult problem due to the absence of the small parameter. This can be illustrated by the example of a coarse-graining procedure, which was extremely successful in engineering turbulence simulations. Consider the wave number k in the inertial range. Let us denote $\mathbf{v}^<(\mathbf{q})$ the Fourier components of the velocity field with all modes $\mathbf{v}^<(\mathbf{q})=0$ when $q>k$. The modes $v(q)$ with $q>k$ are denoted as $v^>(q)$. The coarse-grained field in the physical space is defined then as

$$\mathbf{v}_r(\mathbf{x},t)=\int_{k<1/r}\mathbf{v}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}d^3k.$$

The equation of motion for $v_i^<(k)$ in the Fourier space resembles the Navier-Stokes equation with effective viscosity [10,17]

$$\nu(k)\approx\left(\frac{(d-d_c)(d+\frac{1}{2})}{3.2d(d+2)}\right)^{1/3}\mathcal{E}^{1/3}k^{-4/3}$$

with $d_c\approx 2.56$ plus high-order nonlinearities. The parameter 3.2 in the above relation, evaluated at $d=3$, is in fact a weak function of space dimensionality and $\mathcal{E}=P=O(1)$. This expression is derived assuming close-to-Gaussian statistics of the small-scale turbulence with

$$\overline{v_i(k,\omega)v_j(k',\omega')}\propto\frac{k^{-d}}{-i\omega+\nu(k)k^2}\,\delta(k+k')\,\delta(\omega+\omega'),$$

which is accurate when $d-d_c\to 0$ [10] (see below).

In the physical space the effective viscosity of the coarse-grained field

$$\nu_r\approx(d-d_c)^{1/3}N(\mathcal{E}r^4)^{1/3}\approx v_r^2\tau_r$$

defines the relaxation time τ_r, which is a characteristic time of interaction of the field v_r with the eliminated modes acting on the scales $l<r$.

The difficulty of the theory is in the higher nonlinearities

$$(v_r\nabla)^n\tau_r^{n-1}v_r,$$

and the dimensionless expansion parameter

$$\frac{\tau_r}{\theta_r}\approx\nabla_r v_r\tau_r\approx\frac{\tau_r v_r}{r}$$

is nothing but the ratio of the relaxation time τ_r, strongly influenced by the pressure gradients contributions, to the translational time θ_r, characterizing the tendency of the ''large-scale'' longitudinal velocity fluctuation at the scale r to form a small-scale ''shock'' (atypical in the absence of pressure). In both 2D and 3D these times are of the same order and that is why trancation of the expansion is a very difficult problem.

This is not so in the vicinity of $d=d_c$. Let us assume that the theory can be analytically continued to the noninteger dimensions [9]. Then, since the energy flux, prescribed by the power of the forcing $P=O(1)$, we have

$$\mathcal{E}\approx\frac{\partial}{\partial r}\langle\Delta u(\Delta v)^2\rangle\approx\frac{\partial}{\partial r}\langle(\Delta u)^3\rangle\approx\nu\left\langle\left(\frac{\partial v_{ri}}{\partial r_j}\right)^2\right\rangle=O(1).$$

This means that

$$v_{r,rms}\approx(d-d_c)^{-1/6}(\mathcal{E}r)^{1/3},$$

the dissipation wave number $k_d\approx(d-d_c)^{1/4}(\mathcal{E}/\nu_0^3)^{1/4}$ and, as a consequence,

$$\frac{\tau_r}{\theta_r}\approx(d-d_c)^{1/2},$$

which will serve as a small parameter of the theory when $d-d_c\to 0+$.

These relations tell us that the turbulent intensity grows to infinity as $d-d_c\to 0$ where the energy flux changes its sign. The time needed to reach the steady state is estimated easily,

$$T\approx(d-d_c)^{-1/3}\mathcal{E}^{-1/3}r^{2/3},$$

after which a close-to-Kolmogorov spectrum is expected both above and below d_c. Thus, at $d=d_c$ the the flow is unsteady.

The above results enable one to derive a plausable estimate for the dissipation term D. It is clear from the Navier-Stokes equations that

$$\mathcal{E}=-\tfrac{1}{2}\partial_i v_i v_j^2-\tfrac{1}{2}\partial_t v^2-\partial_i v_i p.$$

The coarsed-grained expression in the low-frequency limit is

$$\mathcal{E}\approx-\frac{1}{2}\frac{\partial}{\partial r_i}v_{ri}v_{rj}^2[1+O(d-d_c)]. \tag{37}$$

To arrive at an expression for D we assume $v_r\approx\Delta v$, which leads to an expression very similar to Kolmogorov's refined similarity hypothesis.

VIII. THREE-DIMENSIONAL FLOW

The most important feature of two-dimensional turbulence, considered in a previous section, is the irrelevance of the dissipation processes in the inverse cascade range when $d<d_c$. It is this irrelevance that was responsible for the Gaussian probability density of transverse velocity differences.

The model for the dissipation contribution D in the limit $\eta_2\to 0$ is readily evaluated from Eq. (37). We would like to keep at least some information about Δv and the expression must be invariant under the transformation $\mathbf{v}\to-\mathbf{v}$ and $\mathbf{x}\to-\mathbf{x}$. In addition, the expression must be local in the physical space. Based on these considerations we have

$$\mathcal{E}_v\approx c(d)\Delta u\Delta v\frac{\partial\Delta v}{\partial r},$$

where $\mathcal{E}_v$ is a dissipation rate of the "v contribution to kinetic energy" $K_v=(1/2)v^2$. The locality of this model is clear since $\partial_r\Delta v=\partial_1 v(x_1)+\partial_2 v(x_2)$. The problem is in the evaluation of the coefficient $c(d)$ since, in principle, it can be singular at $d=d_c$. Indeed, it is clear that the dissipation term is zero at $d\leqslant d_c$. However, the point $d=d_c$ that separates the inverse and direct cascade ranges is a singularity due to infinitely large amplitudes of velocity fluctuations. All we can say at this point is that the PDF can be represented as a sum of even and odd functions of Δv. The symmetric part has a width growing with $d-d_c\to 0+$, while the width of the odd one is $O(1)$. The behavior of $c(d)$ in the vicinity of d_c is not clear. We feel that it is $O(1)$ at $d>d_c$ and zero at $d\leqslant d_c$. Thus, we have

$$D\approx c(d)\eta_3\partial_{\eta_2}\partial_{\eta_3}\partial_r Z\approx c(d)\eta_3^2\left\langle\Delta u\Delta v\frac{\partial\Delta v}{\partial r}e^{\eta_2\Delta u+\eta_3\Delta v}\right\rangle+O(\partial_r\partial_{\eta_2}Z). \tag{38}$$

This expression obeys the basic symmetries of the Navier-Stokes equation.

The last term in the right-hand side of Eq. (38) simply modifies the coefficient in front of the first term in the left-hand side of Eq. (6) and does not generate anything new. The expression (38) resembles Kolmogorov's refined similarity hypothesis, connecting the dissipation rate, averaged over a region of radius r, with $(\Delta u)^3$. Thus, in the limit $\eta_2\to 0$

$$\left[\partial_{\eta_1}\partial_{\eta_2}+\frac{2}{r}\partial_{\eta_2}+(1+h)\frac{\eta_3}{r}\frac{\partial^2}{\partial_{\eta_2}\partial_{\eta_3}}+c(d)\eta_3\partial_{\eta_2}\partial_{\eta_3}\partial_r\right]Z(\eta_2=0,\eta_3,r)=(\eta_3^2+\eta_2^2)\frac{2P}{3}[1-\cos(k_f r)]Z-b\frac{\eta_3}{r^{2/3}}\partial_3 Z. \tag{39}$$

The $d-d_c>0$ counterpart of Eq. (35) is

$$\left[\partial_{\eta_1}+(1+h+b)\frac{\eta_3}{r}\frac{\partial}{\partial\eta_3}+c(d)\eta_3\partial_{\eta_3}\partial_r\right]Z(\eta_2=0,\eta_3,r)=a\frac{2P}{3}\frac{1-\cos(k_f r)}{r^{1/3}}\eta_3^2 Z(0,\eta_3,r), \tag{40}$$

with $\Psi(\eta_3)$ chosen in such a way that the generating function $Z(0,0,r)=1$. We consider two limiting cases.

Small-scale dynamics

Inverse Laplace transform of Eq. (40) without the right side gives an equation for the PDF $P(\Delta v,r)\equiv P(V,r)$,

$$\frac{\partial P}{\partial r}+\frac{1+3\beta}{3r}\frac{\partial}{\partial V}VP-\beta\frac{\partial}{\partial V}V\frac{\partial P}{\partial r}=0, \tag{41}$$

where $\beta\propto c(d)$. Since $S_{0,3}=0$, the coefficients in Eq. (41) are chosen to give $s_{0,3}=\overline{|\Delta v|^3}=a_3 Pr$ with an undetermined amplitude a_3. This is an assumption of the present theory, not based on rigorous theoretical considerations. Seeking a solution in the form $S_{0,n}=\langle(\Delta v)^n\rangle>\propto r^{\xi_n}$ gives

$$\xi_n=\frac{1+3\beta}{3(1+\beta n)}n\approx\frac{1.15}{3(1+0.05n)}n, \tag{42}$$

which was derived in [18] together with $\beta\approx 0.05$. It follows from Eq. (41) that $P(0,r)\propto r^{-\kappa}$, where $\kappa=(1+3\beta)/3(1-\beta)\approx 0.4$ for $\beta=0.05$. Very often the experimental data are presented as $P(X,r)$, where $X=V/r^{\mu}$ with $2\mu=\xi_2\approx 0.696$ for $\beta=0.05$. This gives $P(X=0,r)\propto r^{-\kappa+\mu}\approx r^{-0.052}$ compared with the experimental data by Sreenivasan [19]: $-\kappa+\mu\approx-0.06$.

Let us write $P(V,r)=r^{-\kappa}F(V/r^{\kappa},r)=r^{-\kappa}F(Y,r)$, so that F obeys the following equation:

$$(1-\beta)r\frac{\partial F}{\partial r}+\beta k\frac{\partial}{\partial Y}Y^2\frac{\partial F}{\partial Y}-\beta Yr\frac{\partial^2 F}{\partial Y\partial r}=0. \tag{43}$$

Next, changing the variables again to $-\infty<y=Ln(Y)<\infty$, substituting this into Eq. (43), and evaluating the Fourier transform of the resulting equation, gives

$$(1-\beta)r\frac{\partial F}{\partial r}+\beta\kappa(ik-k^2)F-ik\beta r\frac{\partial F}{\partial r}=0 \tag{44}$$

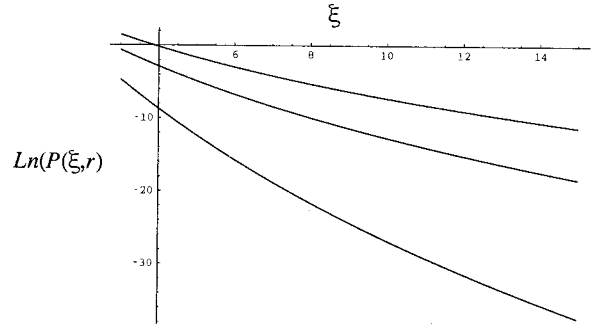

FIG. 1. $Ln[P(\xi,r)]$. From bottom to top: $r/L=0.1$, 0.01, and 0.001, respectively.

with the result $F\propto r^{\gamma(k)}$, where $\gamma(k)=\beta\kappa[(-ik+k^2)/(1-\beta-i\beta k)]Ln(r/L)$ with $r/L\ll 1$. We have to evaluate the inverse Fourier transform

$$F=\int_{-\infty}^{\infty}dk\, e^{-ik_y}e^{\gamma(k)} \tag{45}$$

in the limit $y=O(1)$ and $r\rightarrow 0$ so that $Ln(r/L)\rightarrow-\infty$. The integral can be calculated exactly. However, the resulting expression is very involved. Expanding the denominator of $\gamma(k)$ gives

$$F=\int_{-\infty}^{\infty}dk\,\exp\left[-ik\left(y+\frac{\beta\kappa[Ln(r)]}{1-\beta}\right)\right]\times\exp\left(-\frac{\beta_\kappa(1+\beta)|Ln(r)|}{(1-\beta)}k^2\right) \tag{46}$$

and

$$F\propto\frac{1}{\sqrt{\Omega(r)}}\exp\left(-\frac{[Ln(\xi)]^2}{4\Omega}\right) \tag{47}$$

with $\xi=V/r^{\kappa/(1-\beta)}$ and $\Omega(r)=4\beta\kappa[(1+\beta)/(1-\beta)]|Ln(r/L)|$.

To understand the range of validity of this expression, let us evaluate $\langle V^n\rangle$ using the expression (47) for the PDF. Simple integration and neglecting $O(\beta^2)$ contributions gives: $\langle V^n\rangle\propto r^{\alpha_n}$ with $\alpha_n=(1+3\beta)[n-\beta(n^2+2)]/3$. Comparing this relation with the exact result (42) we conclude that the expression for the PDF, calculated above, is valid in the range $n\gg 1$ and $\beta n\ll 1$. The properties of the PDF in the range $3\leqslant\xi\leqslant 15$ are demonstrated in Fig. 1 for $r/L=0.1,0.01,0.001$. The log-normal distribution (47) is valid in a certain (wide but limited) range of the V variation. It is clear from Eq. (42) that neglecting the dissipation terms $[c(d-d_c)\propto\beta=0]$ leads to $\xi_n=n/3$, i.e., the disappearance of anomalous scaling of moments of velocity differences. This result agrees with the well-developed phenomenology, attributing intermittency to the dissipation rate fluctuations: the stronger the fluctuations, the smaller the fraction of the total space they occupy [8,14]. To the best of our knowledge, this is the first work leading to the multifractal distribution of velocity differences as a result of approximations made directly on the Navier-Stokes equations.

To investigate the probability density $P(Y,r)$ in the limit $Y\rightarrow 0$ we introduce an expansion

$$F(Y,r)=\sum_n C_n Y^{2n}f_{2n}(r). \tag{48}$$

Substituting this into Eq. (44) gives

$$f_{2n}\propto\left(\frac{r}{L}\right)^{-\beta_\kappa 2n(2n-1)/[1-\beta(1+n)]}. \tag{49}$$

It is seen from Eqs. (48) and (49) that the PDF starts bending from the log-normal slope (47) toward $\partial_Y F(Y,r)=0$ at $Y=0$ at

$$Y<\left(\frac{r}{L}\right)^{\beta_\kappa/(1-2\beta)}. \tag{50}$$

This inequality shows that as $r\rightarrow 0$ the PDF develops a narrow cusp at the origin $Y=0$. If the probability density is plotted in the dimensionless variable X, the bending starts at $X\approx r^{0.07}$. This value was calculated, as above, with $\beta=0.05$.

Large-scale limit: $r/L\approx 1$

Now let us investigate the large-scale limit $r/L\rightarrow 1$. Realizing that β can be an $O(1)$ constant, for illustration purposes we will investigate the large-scale limit pretending that $\beta(d)\rightarrow 0$ as $d\rightarrow d_c$. This is also useful since the estimated value of $\beta\approx 0.05$ at $d=3$ is numerically small. In this limit the right-hand side of Eq. (41) is $O(\eta_3^2 Z)$ and cannot be neglected. Repeating the procedure leads to an equation

$$\frac{\partial P}{\partial r}+\frac{1+3\beta}{3r}\frac{\partial}{\partial V}VP-\beta\frac{\partial}{\partial V}V\frac{\partial P}{\partial r}=a\frac{P}{(Pr)^{1/3}}\frac{\partial^2 Z}{\partial V^2}, \tag{51}$$

where a is a proportionality coefficient and $r\approx L$. As one can see from this equation in the limit of small β the solution to this equation approaches Gaussian. It is also clear that for any finite β, the tails of the PDF are strongly non-Gaussian when

$$\beta Y^2\gg 1. \tag{52}$$

This estimate means that, according to the theory presented above, the perturbative treatment of deviations from the mean-field Gaussian theory is possible but it involves two parameters: the ratio $\epsilon_0=1-r/L\ll 1$ and $\beta\ll 1$. The fact that the ''real-life'' $\beta\approx 1/20$ may explain why the experimentally observed PDF of the large-scale ($\epsilon\ll 1$) velocity fluctuations was so close to the Gaussian (see [8] and references therein). It is also seen from Eq. (52) that at $(r/L)^2\approx\beta\approx 0.05$ the PDF is dominated by a Gaussian central part.

IX. CONCLUSIONS

Equation (6) formulates the theory of turbulence in terms of ''only'' two unknowns: pressure and dissipation terms I_p and D, respectively. It provides a mathematical testing ground for various analytic expressions and models obtained from numerical simulations.

Armed with the experimental and numerical data, supporting Gaussian statistics of transverse velocity differenced in two-dimensional flows, we showed that the mean-field approximation [the lowest-order term of the expansion (28)] for the pressure contributions (29) leads to both Kolmogorov scaling and Gaussian statistics of transverse velocity differences. In addition, Eq. (6) shows that the single-point PDF's in 2D turbulence are Gaussian. It is to be stressed that 2D turbulence cannot be a Gaussian process and probability density $P(\Delta u,\Delta v,r)=$ is not a Gaussian. It is only the PDF $P(\Delta v,r)=\int P(\Delta u,\Delta v,r)d\Delta u$ which is a Gaussian. This statement violates no dynamic constraints. If this is so, then transverse velocity differences are a good candidate to serve as an ''order parameter'' of turbulence theory.

One of the most interesting outcomes of the present theory is the discovery of the existence of the two time scales in the system which are very different in the vicinity of $d=d_c$. This difference enables one to coarse grain the Navier-Stokes equations and neglect all high-order nonlinearities generated by the procedure. Using this result the model for for the dissipation term D was derived.

The as yet unresolved ambiguity of this model is its behavior as $d\to d_c$. If the transition from 3D to the nonintermittent state at $d<d_c$ is smooth, then $\beta\to 0$ and the resulting equation shows the onset of both anomalous scaling and non-Gaussian statistics. The transition can be singular, however: right at $d>d_c$ the coefficient β can become $O(1)$ and a weakly intermittent state and weak coupling limit do not exist. In this case, due to the existence of the small parameter, enabling the evaluation of the dissipation expression D, the theory nonperturbatively predicts both the shape of the PDF and scaling exponents provided the small parameter $S_{3,0}/(S_{2,0})^{3/2}\to 0$ as $d\to d_c$. This result is possible since $D=D_0+O(d-d_c)$ avd even in the limit $d\to d_c$, the model $D=O(1)$. At $d-d_c<0$, $D=0$, leading to the Gaussian PDF of transverse velocity differences. Experimental and numerical investigation of hydrodynamics in a noninteger space dimension is impossible. However, it was demonstrated by Jensen [20] that a force-driven shell model yields the changing sign of the energy flux upon variation of a leading parameter. The numerical solution at a critical point (zero flux) demonstrated an unsteady state with the growing total energy and the energy spectrum concentrated in the vicinity of k_f. The calculation also gave the Kolmogorov energy spectrum at ''$d>d_c$'' with a growing Kolmogorov constant as $d\to d_c$. It is not yet clear how the eddy viscosity approximation works for the shell model, but, since the $O(1)$ energy flux is fixed by the forcing function, the growth of kinetic energy must be related to a relaxation time $\tau_r\to 0$ at $d=d_c$ and a corresponding small parameter. This result also shows that the phenomenon is very robust: all one needs is a point at which the energy cascade changes its direction. The results of the shell model investigation will be published elsewhere [21].

Expression (47) is similar to the one obtained in a groundbreaking paper by Polyakov on the scale invariance of strong interactions, where the multifractal scaling and the PDF were analytically derived [22,23]. In the review paper [23] Polyakov noticed that the exact result can be simply reproduced by considering a cascade process with a heavy stream (particle) transformed into lighter streams at each step of the cascade (fission). Due to the relativistic effects the higher the energy of the particle, the smaller the angle of a cone, accessible to the fragments formed as a result of fission. Thus, the larger the number of a cascade step, the smaller is the fraction of space occupied by the particles [23].

The theory presented in this paper describes many experimental observations. Still, an understanding of the limits of validity of expression (29) is crucial for the final assessment of the theory. Relation (32) shows that Eq. (29) is consistent with the Gaussian tails of the PDF. However, at the present stage we are unable to prove that Eq. (29) is the only expression leading to this result. The problem is that without experimental detection of at least some deviations from the Gaussian statistics of transverse velocity differences, one will not be able to understand the limits of validity of Eq. (29). Given the state-of-the-art of numerical simulations, this goal may not be that simple.

ACKNOWLEDGMENTS

I am grateful to A. Polyakov for the most interesting and stimulating discussions. Help from K. R. Sreenivasan and M. Vergassola, who shared with me their sometimes unpublished experimental data, was most useful. The recent numerical results of M. Jensen produced not only support for the conclusions of this paper but also demonstrated a surprising power and importance of a shell model. The recent derivation by R. Hill of some of the relations of Sec. III, using a totally different method, was most illuminating [24]. I also benefited from conversations with B. Shraiman, P. Constantin, C. Doering, T. Gotoh, and T. Kambe.

[1] M. Cherkov, G. Falkovich, I. Kolokolov, and V. Lebedev, Phys. Rev. E **52**, 4924 (1995).
[2] K. Gawedzki and A. Kupiainen, Phys. Rev. Lett. **75**, 3834 (1995).
[3] B. I. Shraiman and E. D. Siggia, C. R. Acad. Sci., Ser. IIb: Mec., Phys., Chim., Astron. **321**, 279 (1995).
[4] L. Smith and V. Yakhot, J. Fluid Mech. **274**, 115 (1994).
[5] L. Smith and V. Yakhot, Phys. Rev. Lett. **71**, 352 (1993).
[6] P. Tabeling and J. Paret, Phys. Fluids **12**, 3126 (1998).

[7] G. Boffetta, A. Celani, and M. Vergassola, Phys. Rev. E **61**, 29 (2000).
[8] U. Frisch, *Turbulence* (Cambridge University Press, Cambridge, 1995).
[9] U. Frisch and J. D. Fournier, Phys. Rev. A **17**, 747 (1978).
[10] V. Yakhot and S. A. Orszag, Phys. Rev. Lett. **57**, 1722 (1986).
[11] V. Yakhot, Phys. Rev. E **60**, 5544 (1999).
[12] A. M. Polyakov, Phys. Rev. E **52**, 6183 (1995).
[13] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, Oxford, 1998), p. 14.
[14] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics* (MIT Press, Cambridge, 1971), Vol. 1.
[15] Ya. G. Sinai and V. Yakhot, Phys. Rev. Lett. **63**, 1962 (1989).
[16] R. H. Kraichnan, Phys. Fluids **10**, 1417 (1967).
[17] Yakhot Orszag, J. Sci. Comput. **1**, 3 (1986).
[18] V. Yakhot, Phys. Rev. E **55**, 329 (1997); **57**, 1737 (1998).
[19] K. R. Sreenivasan (private communication).
[20] M. Jensen (private communication).
[21] M. Jensen, P. Guliani, and V. Yakhot (unpublished).
[22] A. M. Polyakov, Sov. Phys. JETP **34**, 1177 (1972).
[23] A. M. Polyakov (unpublished).
[24] R. Hill (private communication).